Complex Numbers, Time Functions, Vectors and Fourier Series

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## Contents

0.1 ABSTRACT ................................................................. iii

1 INTRODUCTION AND THEORY ........................................... 1
  1.1 INTRODUCTION .......................................................... 1
  1.2 BASIC THEORY ......................................................... 1
    1.2.1 Time Functions, Complex Plane, Periodic Functions .......... 1
    1.2.2 Complex Exponents .............................................. 10
    1.2.3 De Moivre’s Theorem ............................................ 12
    1.2.4 Vectors and Vector Spaces ..................................... 14
      1.2.4.1 Cross Product ............................................ 16
      1.2.4.2 Inner Product ............................................ 17
        1.2.4.2.1 Properties of Inner Product ......................... 18
    1.2.5 Vector Space .................................................... 18
      1.2.5.1 Subspace ................................................ 20
    1.2.6 Basis ............................................................ 20
    1.2.7 Orthogonal and Orthonormal Sets ............................... 21

2 FOURIER SERIES ...................................................... 23
2.1 DEFINITION OF FOURIER SERIES .......................... 23
2.1.1 Fourier Coefficients ................................. 26
ABSTRACT

Fourier series and transform are perhaps one of the most significant tools in Electrical and Computer Engineering field. Yet, the subject is very hard to grab for students, even at graduate level. Therefore, in This study we investigated roots of unity, vector spaces and Fourier series. Despite the fact that we constructed regular polygons by using Fourier series, we also stressed the fact that any time signal $f(t)$ that is periodic with a period of $T_0$, has Fourier series which can be considered as a vector space, formed by basis vectors that are roots of unity.
List of Figures

1.1 Graphical representation of a complex number in the complex plane . . . . 2
1.2 Cosine and sine functions plotted versus time . . . . . . . . . . . . . . . . 3
1.3 Cosine and sine functions plotted versus phase . . . . . . . . . . . . . . . . 4
1.4 Sine wave plotted versus cosine wave . . . . . . . . . . . . . . . . . . . . . 10
1.5 A vector in two dimensional space and its components (coefficients) . . . . 15
Chapter 1

INTRODUCTION AND THEORY

INTRODUCTION

Complex numbers have always been of great use, not only because of being the superset for real numbers, but very helpful in all areas of science, especially in engineering.

BASIC THEORY

Time Functions, Complex Plane, Periodic Functions

Consider the case of acoustic or electrical signals called cosine or sine signals (waves). Mathematically it is represented as

\[ x(t) = A \cos(\omega_o + \phi) \]  

(1.2.1)

Where

\( A \) is the amplitude
$w_o$ is the angular (radian) frequency \((rad/sec)\) and $w_o = 2\pi f_o$ with $f_o$ being the cyclic frequency, having the unit of \(Cycles/Second\) or \(Hertz\).

$\phi$ is called phaseshift, representing how much the signal $x(t)$ shifted from its actual phase, which is $w_o$.

$x(t)$, of course for this specific example, is a continuous function of time $t$.

We are going to show four out of five ways described in [1] to represent complex numbers.

We can graphically represent a complex number as in Figure 1.1.

![Figure 1.1: Graphical representation of a complex number in the complex plane.](image-url)
We know from trigonometry that $\sin \theta = \frac{y}{r}$ and then $y = r \sin \theta$. Similarly, $\cos \theta = \frac{x}{r}$ and then $x = r \cos \theta$.

In Figure 1.2 we plot two functions, $\sin 2\pi t$ and $\cos 2\pi t$. Following our notation above, one should notice that the cyclic frequency for both these functions is $f_0 = 1$ Hertz (Hz).

We know that $\sin^2 \theta + \cos^2 \theta = 1$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$. Using these two identities, the following identities can be derived easily;

\[
\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \\
\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)
\]

(1.2.2)

Figure 1.2: Cosine and sine functions plotted versus time
As can be observed from the above graph sine and cosine waves are closely related. We
know that derivative of \( \sin \theta \) is equal to \( \cos \theta \) and the derivative of \( \cos \theta \), \( \frac{d\cos \theta}{dt} = - \sin \theta \). That is to say, the cosine function is the slope of the sine function and sine function is the negative slope of the cosine function. Looking at Figure 1.2 we can show that cosine function will overlap sine function if we shift cosine by 0.25 (0.25) seconds. The signal drawn with a solid line in Figure 1.2 is \( \cos 2\pi t \). Considering the general form of this function being \( \cos \theta \), we can say that \( \theta = 2\pi t \). Notice again that, the cyclic frequency for this function is 1 Hz. If we substitute \( t = 0.25 \) second into \( \theta = 2\pi t \), we get \( \theta = \frac{\pi}{2} \). What we did now was to phase shift the cosine function. Hence, we can say that the phase shift between sine and cosine functions is \( \frac{\pi}{2} \) radians/second. We can see this more clearly from Figure 1.3.

![Figure 1.3: Cosine and sine functions plotted versus phase](image)

Figure 1.3: Cosine and sine functions plotted versus phase
Phase shifting cosine wave will produce sine wave and vice versa. Because of the fact that sine and cosine functions are periodic (repeating the same waveform), we can use equation 1.2.1 to show that phase shifted cosine function equals to sine function or vice and versa,

\[ x(t) = A \cos(\omega_0 t + \phi) = A \sin(\omega_0 t + \phi + \frac{\pi}{2}) \]

\[ x(t) = A \cos(\omega_0 t + \phi + \frac{\pi}{2}) = A \sin(\omega_0 t + \phi) \] (1.2.3)

The amplitude \( A \) is such that the function oscillates between \(-A\) and \( A \). Therefore, Equation 1.2.1 can be written in terms of either sine or cosine, and we prefer to express it with cosine, because cosine is an even function, meaning the function is symmetric with respect to the axis belonging to its range (in this case the vertical axis), or algebraically, \( \cos \theta = \cos(-\theta) \). Now that we gave the simple definition of an even function, let us give the definition for an odd function too. A function \( y(t) \) is said to be odd if it is symmetric with respect to the origin (in this case \((0,0)\) point), or algebraically speaking, \( y(t) = -y(-t) \). The simplest example right now probably would be the sine function, since \( \sin \theta = -\sin(-\theta) \). A function that does not satisfy any of these condition is called neither even nor odd.

A periodic signal has the cyclic frequency \( f_o = \frac{1}{T_o} \) cycles per second; where \( T_o \) is referred to as period of the function, the required time to complete a full cycle. Therefore, \( f_o \) is a legitimate, and appropriate measuring unit when dealing with the radial frequency \( \omega_o \), which is radians per second.

In Figure 1.1, the complex number is represented by \( z = (x,y) \); where \( x = Re\{z\} \) is the real part and \( y = Im\{z\} \) is the imaginary part. The imaginary part of the complex number may be represented with a prefix \( j \), with \( j = \sqrt{-1} \). We can then write the complex number in Cartesian form as \( z = x + jy \). Looking at Figure 1.1, we can see that complex numbers
are represented as a point \((z)\) in complex plane, having their real and imaginary parts as coordinates and these points in complex plane are analogous to vectors in two-dimensional space. Since vectors have length and directions, Cartesian form of complex numbers can be also represented as the polar form, as, \(z = re^{i\theta}\). Vector has a length of \(r\) and makes an angle of \(\theta\) with the real axis in counterclockwise direction. If the angle was to be in clockwise direction, then it would have a minus sign.

We convert the Cartesian form to polar form so that we can use the trigonometry and the Pythagorean theorem to compute Cartesian coordinates \((x, y)\) from the polar variables \(z = re^{i\theta}\). By using Pythagorean theorem, we can compute \(x, y, \text{ and } r\); \(x = r\cos\theta\), and \(y = r\sin\theta\), where \(r = \sqrt{x^2 + y^2}\). Notice again, that this way or representing a complex number is equivalent to representing a point in two dimensional space. This is one of the ways to represent a complex number.

Another way to look at the complex number is by ordered pairs, which was first done by Hamilton. Consider two complex numbers, \(z_1 = x_1 + jy_1\) and \(z_2 = x_2 + jy_2\). We represent them as \((x_1, y_1)\) and \((x_2, y_2)\). For these two ordered pairs to be equal, the corresponding numbers in both pairs have to be equal. Namely, if \((x_1, y_1) = (x_2, y_2)\) then \(x_1 = x_2\) and \(y_1 = y_2\). The addition is defined as \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\), while the multiplication is \((x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\). This operation assumes that the complex operator \(j\) is “hidden” with the second element of every ordered pair. The subtraction is rather trivial while division is a little more complicated than multiplication. \(\frac{(x_1, y_1)}{(x_2, y_2)} = (x_1, y_1) \times \frac{1}{(x_2, y_2)}\), where \(\frac{1}{(x_2, y_2)} = \left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right)\). We can justify this last equality by taking the product \((x_2, y_2) \times \frac{1}{(x_2, y_2)}\) which equals to \(\left(\frac{x_2x_2 - y_2(-y_2)}{x_2^2 + y_2^2}, \frac{x_2y_2 + y_2x_2}{x_2^2 + y_2^2}\right)\). Looking carefully, we see that the result of the previous operation is \((1, 0)\).

The third way to look at complex numbers is rather interesting.
Let us consider the matrix \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]. If we square this matrix, by using matrix algebra, we get
\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\], where as we know from matrix algebra, \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\] is unity matrix of size 2. Hence, we can say that \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\].

Remembering our very first representation of complex numbers, which was \(z = x + jy\), we can write
\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
+ \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

hence \[
\begin{pmatrix}
x + 0 \\
0 + y
\end{pmatrix}
= \begin{pmatrix}
x + 0 \\
0 + y
\end{pmatrix}
\]

finally \[
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
x \\
y
\end{pmatrix}
\].

The last way we are going to explain is based on using polynomials that have real coefficients; namely, \(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\) where \(a_k = 0 \cdots n \in \mathbb{R}\), and polynomial is said to have, or is said to be of degree \(n\). We then divide the polynomials with \(x^2 + 1\), because \(j\) is a root of this divider. Of course, after the division operation is complete, we will end up with a quotient and a remainder of degree less than degree of divisor, in this case either 1 or 0, having the form \(a_1x + a_0\). If none of the constants equal to zero, we will have a first order polynomial as a remainder. If the coefficient \(a_1\) is zero, then we have a zero order polynomial, which is a constant. Moreover, if \(a_0\) is zero in the case of zero order polynomial, and we say the remainder is zero.

Two polynomials, regardless of their degrees, are said to be equivalent when they have the same remainder as a result of the division by \(x^2 + 1\). Hence, we will have more than one polynomial falling into same class, hence, we will have what we call *equivalence classes*.
appearing. An equivalence class is also called a sack in which we will have many of the same element (rather different than definition of a set). For example, we may have many 1’s as remainders of the corresponding polygons divided by \(x^2 + 1\), the set of polynomials \(1, x^2 + 2, 2(x^2 + 1) + 1, \ldots n(x^2 + 1) + 1\ldots\). We can give the same example for the equivalence class 0 as well.

Examining a little closely, we see that what we do here corresponds to what’s called as modular arithmetic. More specifically, what we do is to take the “mod \((x^2 + 1)\)” of the polynomials. Since \(x^2 + 1\) is equivalent to zero, we can say that \(x^2 + 1 = 0\) and hence \(x^2 = -1\). Now we can say that we have a simple polynomial \(x\) that equals to -1 when squared. In another words this is the polynomial of the form \(a_1x + a_0\), in which \(a_1 = 1\) and \(a_0 = 0\). This conclusion of course brings up the fact that the multiplicative inverse of the class of polynomials with remainder \(a_1x + a_0\), is the class of polynomials with remainder \(-a_1x+a_0\)

\[
\frac{-a_1x+a_0}{a_1^2+a_0^2}.
\]

Notice again that what we meant was \(x^2 = -1 \mod (x^2 + 1)\), in another words, \(x^2\) is congruent (equivalent) to -1 and hence \(x = \sqrt{-1}\).

The important conclusion from this approach is that the equivalence classes formed after dividing set of polynomials with real coefficients with \(x^2 + 1\) are in one to one correspondence with complex numbers, hence they are isomorphic.

If we summarize, we looked at the complex numbers from 4 different point of view; namely

1. A point in the 2-dimensional space (cartesian coordinates, \(x, y\) plane)

2. Ordered pairs of real numbers, with the complex operator \(j\) hidden in the second element of each pair
3. $z_{2 \times 2}$ matrices with real elements

4. Equivalence classes of polynomials with real coefficients.

Looking back at the polar form of complex numbers that we defined from cartesian coordinates; rather than using $z = re^{j\theta}$, we can use another polar form provided by Euler for the complex exponential; $e^{j\theta} = \cos \theta + j \sin \theta$. Thus, the Cartesian pair $\cos \theta$ and $\sin \theta$ represents any point on a circle with radius of 1. Writing Euler’s representation for a more general case, we have $z = re^{j\theta} = r \cos \theta + jr \sin \theta$. In this manner, we write a complex exponential signal as

$$f(t) = Ae^{j(w_o t + \phi)}$$  \hspace{1cm} (1.2.4)

In equation 1.2.4 $f(t)$ is a complex valued function of time $t$. Magnitude of the function is $|f(t)| = A$ and the angle is $\arg\{f(t)\} = w_o t + \phi$ which corresponds to $\theta$. Now, it is clear that $\theta$ is a function of time.

With the help of Euler’s $e^{j\theta} = \cos \theta + j \sin \theta$, the complex number can be shown in the Cartesian form as

$$f(t) = Ae^{j(w_o t + \phi)} = A \cos(w_o t + \phi) + jA \sin(w_o t + \phi)$$  \hspace{1cm} (1.2.5)

We are now able to plot the real part of the function versus the imaginary part and observe the plot when the phase shift is $\frac{\pi}{2}$ radians. Basically, what we are doing by this plot is rotating the phasor in the complex plane.
Figure 1.4: Sine wave plotted versus cosine wave

**Complex Exponents**

Given any complex number $z = Ae^{j\theta}$, and any positive integer $n$, $z^n$ can be expressed as $z^n = A^n e^{jn\theta}$. Similarly, for any positive integer $n$, the $n$ complex roots of the equation (fundamental theorem of algebra states that any $n$-th order equation will have exactly $n$ roots) $\Omega^n = z$ may be expressed in the form

$$z^{1/n} = A^{1/n} e^{j(\theta+2\pi k)/n}$$

where $k = 0, 1, 2, \ldots, n - 1$ \hspace{1cm} (1.2.6)

Using the fact that $\theta$ is determined within a multiple of $2\pi$, the values of $k = n, n + 1, \ldots$ are not used because both sine and cosine are periodic with $2\pi$. The values of sine and
cosine for the argument \((\theta + 2\pi k)\) will have be equal at \(k = n\) and \(k = 0\); \(k = n + 1\) and \(k = 1\), etc.

The function \(z^n\) has a unique value for any positive integer \(n\). The function \(z^{1/n}\) may take on any one of \(n\) values in the complex number system. When \(z\) is a real number, these possible values of the function \(z^{1/n}\) include the real value represented by \(z^{1/n}\) in the set of real numbers. This value is called the principle value of \(z^{1/n}\) in the set of complex numbers. The principle value of \(z^{1/n}\) may be obtained by

1. taking \(k = 0\) when \(z\) is positive, hence Equation 1.2.6 becomes

\[
\frac{z^{1/n}}{A^{1/n}} = e^{j(\theta + 2\pi k)/n} = A^{1/n}e^{j\theta/n}
\]

2. and \(k = (n - 1)/2\) when \(z\) is negative and \(n\) is odd, hence, the result from Equation 1.2.6 is

\[
\frac{z^{1/n}}{A^{1/n}} = e^{j(\theta + 2\pi(n-1)/2)/n} = A^{1/n}e^{j(\theta + \pi(n-1))/n}
\]

3. When \(z\) is negative and \(n\) is even, we define the principle value of \(z^{1/n}\) to be that obtained when \(k = 0\).

4. We shall not attempt to designate principal values of \(z^n\) when \(z\) is imaginary.
De Moivre’s Theorem

If \( n \) is any positive integer and \( z = A(\cos \theta + j \sin \theta) \), then

\[
z^n = [A(\cos \theta + j \sin \theta)]^n = A^n (\cos n\theta + j \sin n\theta) = A^n e^{j n\theta};
\]

\[
z^{1/n} = A^{1/n} \{\cos[(\theta + 2\pi k)/n] + j \sin[(\theta + 2\pi k)/n]\}
\]

\[
= A^{1/n} e^{j(\theta + 2\pi k)/n} \quad \text{where} \quad k = 0, 1, 2, ..., n - 1 \tag{1.2.7}
\]

Where \( A \) is the magnitude (radius of the circle), and \( \theta \) is the angle.

We use this theorem for any complex number \( z = e^{j\theta} \) and positive integer \( n \) to express the unique complex number \( z^n \) and the \( n \) complex roots of the equation \( \Omega^n = z \). Each and every one of these \( n \) roots has the magnitude of \( A^{1/n} \). This of course, means that these \( n \) roots lie on a circle of radius \( A^{1/n} \) that is centered at the origin. Each of the \( n \) roots correspond to a point on this circle that has radius \( A^{1/n} \). These points are equally spaced on the circle, hence the arc length inbetween any consecutive two points is equal. When inspected with respect to their corresponding values of \( k \), their angles (phases, not the magnitudes!) differ by consecutive multiples of \( 2\pi/n \), this conclusion arises from \( e^{j(\theta + 2\pi k)/n} = e^{j\theta/n} e^{j2\pi k/n} \) hence the term \( e^{j2\pi k/n} \) causes the phase difference between the roots (notice that the phase, or angle difference defines the locations of the roots, on the circle of radius \( A^{1/n} \)).

Theorem 1.1

Any complex number \( z = Ae^{j\theta} = A(\cos \theta + j \sin \theta) \neq 0 \) has exactly \( n \) distinct complex roots which as we just saw by De Moivre’s Theorem, can be represented by \( n \) points that are equally spaced on a circle which is centered at origin and has the radius \( A^{1/n} \).
Let’s think on the case when $z = 1$. The roots of this equation (also called cube roots) satisfy $\Omega^3 = 1 = e^{i0} = 1(\cos 0 + \sin 0)$. Therefore, the cube roots of unity may be expressed as

$$\Omega_k = 1^{1/3} e^{j(0 + 2\pi k)/3} = 1^{1/3} \{\cos \left[(0 + 2\pi k)/3\right] + j \sin \left[(0 + 2\pi k)/3\right]\} \quad \text{for } k = 0, 1, 2;$$

or, more explicitly,

$$\Omega_1 = 1(\cos 0 + j \sin 0) = 1 \quad (k = 0),$$
$$\Omega_2 = 1(\cos 2\pi/3 + j \sin 2\pi/3) = -1/2 + j\sqrt{3}/2 \quad (k = 1),$$
$$\Omega_3 = 1(\cos 4\pi/3 + j \sin 4\pi/3) = -1/2 - j\sqrt{3}/2 \quad (k = 2).$$

We call these roots as called the cube roots of unity because of the equation $\Omega^3 = z = 1$; or in another form of representation $z^{1/3} = 1$. For generalization purposes, we need to note that roots for $\Omega^n = z = 1$ or $z^{1/n} = 1$ are called the n-th roots of unity.

The points that represent $\Omega_1$, $\Omega_2$, and $\Omega_3$ on the circle centered at origin and has the radius $A^{1/n}$ are vertices of an equilateral triangle inscribed in a unit circle ($A = 1$) about the origin and having one vertex at $(1,0)$ on the positive $x$-axis. In general, the $n$-th roots of unity can be represented by the vertices of a regular polygon of $n$ vertices (and sides), inscribed in the unit circle centered at the origin, one vertex of the inscribed regular polygon lying at $(1,0)$ on the positive $x$-axis, in other words on the real axis.

Looking from a different point of view, the $n$-th roots of unity form a group of $n$ elements. Moreover, this group is called as a cyclic group because every element of the group can be expressed in terms of a single element hence the group can be represented as a function of one single element that belongs to the group; for $\Omega^3 = 1$, one can represent the three roots of unity as $\Omega_2$, $\Omega_2^2$, $\Omega_2^3$, or as $\Omega_3$, $\Omega_3^2$, $\Omega_3^3$. Let $s$ be an $n$-th root of unity, then we say $s$ is a primitive $n$-th root of unity if $n$ is the smallest positive integer $m$ such that $s^m = 1$, or $[s^m]_{m=n} = 1$. This in group theory means that the primitive $n$-th roots are those of order $n$. 

13
The $n$-th roots of unity, obtained from $z^n = \cos 0 + j \sin 0 = 1$ by De Moivre’s Theorem, are $\cos 2\pi k/n + j \sin 2\pi k/n$ for $k = 0, 1, 2, ..., n - 1$. Consider the root corresponding to $k = 1$, which is $\Omega_1 = \cos 2\pi/n + j \sin 2\pi/n$ and is a primitive $n$-th root of unity, since $\Omega_1^n = \cos 2m\pi/n + j \sin 2m\pi/n$ can equal unity only if $m$ is a multiple of $n$, taking the imaginary part to zero and making the cosine of even number of $\pi$ be equal to unity. Hence, $n$ is the least positive power of $\Omega_1$ that makes it equal to unity. Hence for any positive integer $n$, there exists at least one primitive $n$-th root of unity such that $\Omega^n = 1$. Remember that not all $n$-th roots of unity are primitive $n$-th roots of unity.

Let $s$ be a primitive $n$-th root of unity and $m$ an integer, then $(s^m)^n = (s^n)^m = 1^m = 1$. So, any positive integral power of a primitive $n$-th root of unity is also an $n$-th root of unity (not primitive!). Assume we have $s^m = s^l$, one may assume that $l \leq m$ and then write $s^{m-l} = 1$. Since we conditioned $s$ be a primitive $n$-th root of unity, $n$ is the least positive power of $s$ that makes it equal to unity. Thus, either $m - l$ or $m = l$ must be a multiple of $n$, hence $s^m = s^l$ if and only if $m = l + kn$ for some integer $k$.

**Theorem 1.2**

If $s$ is a primitive $n$-th root of unity, all $n$-th roots of unity are given by the sequence $s, s^2, s^3, ..., s^{n-1}, s^n = 1$. Because $n$-th roots of unity form a cyclic group.

The primitive $n$-th roots of unity are precisely the numbers $s^m$, where $m$ can be any number that is relatively prime to $n$ where $s$ is any primitive $n$-th root of unity [2].

**Vectors and Vector Spaces**

Anybody who worked with linear algebra has dealt with vectors. Typically, engineering students are very familiar with vectors in two and three dimensions. We can refer to Figure 1.5 to see how one can represent a vector in two dimensional space. We could also
represent this vector by its magnitude, or in another words, its norm and direction. Another form of representing this vector would be showing weights of its components on each of the basis vectors that form the space the vector in question lies in. Note that we have one basis vector on each axis of the space the vector lies within. If we consider the two dimensional case, we have the basis vectors (in this case having unity norms) \( \mathbf{u}_x \) and \( \mathbf{u}_y \) that lie on \( x \) and \( y \) axes respectively. Now, by the very idea of superposition, we can represent our vector by means of unit vectors.

\[
\mathbf{v} = a\mathbf{u}_x + b\mathbf{u}_y.
\]

The coefficients \( a \) and \( b \) can be used to represent the vector, keeping in mind the basis vectors;
\[ \vec{v} = [a, b] \text{ or } \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \] these two notations are called the row vector and column vector. We will be using the row vector as our default representation throughout this study. Hence, we will know that

\[ \vec{v} = [a, b, c] \text{ is a vector in three dimensional space } (x, y, z) \text{ and } \vec{v} = [a, b, c, \ldots, n] \text{ is an } n\text{-dimensional vector.} \]

To add or substruct two vectors of the same dimension, we perform the operation on corresponding components. Let \[ \vec{v}_1 = [a_1, b_1] \text{ and } \vec{v}_2 = [a_2, b_2], \text{ then} \]

\[ \vec{v}_r = \vec{v}_1 \pm \vec{v}_2 = [a_1 \pm a_2, b_1 \pm b_2] \]

Let \( \alpha \) be a constant (real or complex), then \( \alpha \vec{v} = [\alpha a, \alpha b] \). This is called scalar multiplication.

Now let us talk about vector (cross) product and dot (inner) product.

**Cross Product**

Let \( \vec{v}_1 = [a_1, b_1] \) and \( \vec{v}_2 = [a_2, b_2] \); then

\[
\vec{v}_r = \vec{v}_1 \times \vec{v}_2 \\
= a_1 a_2 \vec{u}_x \times \vec{u}_x + a_1 b_2 \vec{u}_x \times \vec{u}_y + b_1 a_2 \vec{u}_y \times \vec{u}_x + b_1 b_2 \vec{u}_y \times \vec{u}_y \\
= a_1 b_2 \vec{u}_z - b_1 a_2 \vec{u}_z \\
= (a_1 b_2 - b_1 a_2) \vec{u}_z
\]

We will not go into details of vector product but say that the resulting vector is perpendicular to the place the vectors lie in. Magnitude of the resulting vector can alternatively be determined as both vectors’ magnitudes multiplied, and finally multiplied with sine of the
angle between the vectors. Mathematically representing:

\[ \| \vec{v}_r \| = \| \vec{v}_1 \| \| \vec{v}_2 \| \sin \theta \]

where \( \theta \) is the angle between the vectors \( \vec{v}_1 \) and \( \vec{v}_2 \).

**Inner Product**

We will represent the inner product of vectors for two dimensions and leave the extension to \( n \)-dimensional space to the reader if necessary.

Assume we have two vectors \( \vec{v}_1 = [a_1, b_1] \) and \( \vec{v}_2 = [a_2, b_2] \); the inner product of these two vectors is defined as;

\[
\begin{align*}
v_r &= \vec{v}_1 \cdot \vec{v}_2 \\
&= a_1 a_2 \| \vec{u}_x \| \| \vec{u}_x \| \cos(0) + a_1 b_2 \| \vec{u}_x \| \| \vec{u}_y \| \cos(90) \\
&\quad + b_1 a_2 \| \vec{u}_x \| \| \vec{u}_y \| \cos(90) + b_1 b_2 \| \vec{u}_y \| \| \vec{u}_y \| \cos(0) \\
&= a_1 a_2 + b_1 b_2
\end{align*}
\]

Notice that when you take inner product of two vector, you simply multiply the components at the same bases and sum the whole result up. The result is a scalar and not a vector. This corresponds to projection of a vector on another, and it can be calculated alternatively as

\[ v_r = \| \vec{v}_1 \| \| \vec{v}_2 \| \cos \theta \]

where \( \theta \) is the angle between the vectors \( \vec{v}_1 \) and \( \vec{v}_2 \).

Since we have covered the inner product now, we can benefit from it. When two vectors are orthogonal (perpendicular to one another), their inner product is zero. Now, let us assume a set \( S \) of vectors. If every vector in set \( S \) is orthogonal to each of the vectors in the set but itself, then the set \( S \) is called an orthogonal vector set.
Inner product of a vector $\mathbf{v}$ and a unit vector $\mathbf{u}_1$ from an orthogonal basis set gives the component of vector $\mathbf{v}$ along the unit vector $\mathbf{u}_1$, in another words, it would give the component of vector $\mathbf{v}$ that lies on the axis belonging the unit vector $\mathbf{u}_1$ of the orthogonal basis set. For sake of showing this fact mathematically, let us consider a vector $\mathbf{v} = [a, b]$. Since our vector is a two dimensional one, we know that we will have two base vectors $\mathbf{u}_1 = [1, 0]$, and $\mathbf{u}_2 = [0, 1]$. Then,

$$\mathbf{v} \cdot \mathbf{u}_1 = a1 + b0 = a$$

and

$$\mathbf{v} \cdot \mathbf{u}_2 = a0 + b1 = b$$

Properties of Inner Product

1. $\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{v}_2 \cdot \mathbf{v}_1)^*$ where $*$ stands for complex conjugate.

   and also remember that; $\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{k=1}^{N} v_1(k)v_2^*(k)$

2. $\mathbf{v}_1 \cdot (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3$

3. $(\alpha \mathbf{v}_1) \cdot \mathbf{v}_2 = \alpha (\mathbf{v}_1 \cdot \mathbf{v}_2)$

4. $\mathbf{v}_1 \cdot \mathbf{v}_1 \geq 0; \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = 0$ only if $\mathbf{v}_1 = \mathbf{0}$.

   $\sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = ||\mathbf{v}_1||$ is norm of $\mathbf{v}_1$.

Vector Space

Consider a set of vectors on which vector addition and scalar multiplication (scaling) is defined. After any number of these operations performed on one or more vectors from the set, if the resulting vectors are still in this set, then we call this set as a vector space.
Now, let us list the axioms that must be satisfied so that a set of vectors $V$ can be a vector space.

1. **Commutativity**
   
   $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$

2. **Associativity**
   
   $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$
   
   similarly,
   
   $(\mathbf{v}_1 \cdot \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot (\mathbf{v}_2 \cdot \mathbf{v}_3)$
   
   and finally, for $\alpha, \beta$ constants, be it real or complex, we have
   
   $(\alpha\beta) \mathbf{v} = \alpha(\beta \mathbf{v})$

3. **Additive identity**
   
   $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$ in $V$.

4. **Distributivity**
   
   $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2$
   
   $(\alpha + \beta) \mathbf{v}_1 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_1$

5. **1** $\mathbf{v} = \mathbf{v}$ and $0 \mathbf{v} = \mathbf{0}$

6. **Additive inverse**

   for every vector $\mathbf{v}_1$, there is a vector $\mathbf{v}_2$ such that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ where $\mathbf{v}_2 = -\mathbf{v}_1$.

We could give set of real numbers or set of complex numbers as examples of vector spaces. Another example that might be of special interest for electrical engineers is the set of functions with finite energy, that is $\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$.
**Subspace**

If a subset of vector space \( V \) satisfies all of the axioms to be a space and the resulting from the space axioms are still a member of this subset, then the subset \( S \) is called a *subspace* of \( V \).

A simple example for a subspace would be the set of real numbers being a subspace of complex numbers’ space.

**Theorem 1.3**

A set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) is said to be linearly independent if and only if \( \sum_{i=1}^{N} \alpha_i \mathbf{v}_i = \theta \) is possible only for \( \alpha_i = 0 \) for \( i = 1, 2, \ldots, N \). Note that this means no vector in \( V \) can be written as a linear combination of others that are in \( V \).

**Basis**

We can create a vector space by taking all possible linear combinations of vectors in a set of vectors that are linealy independent. This set of linearly independent vectors is called the *basis* of the vector space. The basis set, therefore, contains the smallest number of linearly independent vectors required to show each element in the vector space \( V \). Of course more than one set can be a basis for a given set (i.e. can satisfy the necessary rules to be a basis set for the vector space in question).

**Example**

Assume a vector space \( V \) having \( \mathbf{v}_i = [a_i, b_i] \) where \( a_i, b_i \in \mathbb{R} \).

Then, any two vectors that are linearly independent can be a basis for the vector space \( V \). Several examples of a basis sets for \( V \) are:
\( V_b = \{[1, -1], [0, 1]\}; \) or \( V_b = \{[0.1, 2], [1, 1]\}; \) etc.

One must remember that, for example;

\( V_b = \{[1, 1], [2, 2]\} \) would not be a basis set since \( [2, 2] = 2[1, 1], \) which tells us that
the 'base vectors' are not linearly independent!

Dimension of the space is equal to number of basis vectors in the basis set, to construct
the vector space. In the case of vector space of real numbers, the dimension (as you can
see from the previous example) is two. If we consider the space of continuous functions
that have finite energy, infinitely many basis vectors would be needed. i.e. this space has
infinite dimensions.

**Orthogonal and Orthonormal Sets**

Two vectors are named to be orthogonal if their inner product is zero. If the basis set is
chosen to be orthogonal and having norms of unity (i.e. orthogonal and unit basis vectors)
then this basis set is called an orthonormal basis set.

As we did before, we can find out the components of a vector by taking the inner
product of it with the orthogonal basis set of the vector space.

Now, let us assume an \( n \)-dimensional vector space \( S \), or more precisely, \( S_n \) with an
orthonormal basis set \( \{ \vec{w}_i \}_{i=1}^{n} \). A vector \( \vec{v} \) in \( S_n \) can of course be represented as linear
combination of the basis vectors, such as;

\[
\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{w}_i
\]

To find any of the coefficients \( \alpha_i \), what we need to do is to take the inner product of \( \vec{v} \)
with each of the basis vectors \( \vec{w}_i \); hence
\[
\{ \alpha_i \}_{i=1}^n = \{ \nabla \cdot \vec{u}_k \}_{k=1}^n \\
= \left\{ \sum_{i=1}^n (\alpha_i \vec{u}_i) \cdot \vec{u}_k k \right\}_{k=1}^n
\]

Hence,

\[
\{ \alpha_i \}_{i=1}^n = \left\{ \sum_{i=1}^n \alpha_i (\vec{u}_i \cdot \vec{u}_k) \right\}_{k=1}^n
\]

Remember that the basis vectors are orthogonal, in this case they are orthonormal, hence;

\[
\vec{u}_i \cdot \vec{u}_k = \begin{cases} 
1 & i = k \\
0 & \text{otherwise}
\end{cases}
\]

So, we conclude that

\[
\alpha_k = \nabla \cdot \vec{u}_k
\]
Chapter 2

FOURIER SERIES

DEFINITION OF FOURIER SERIES

Now we will begin talking about Fourier series, as it is done customarily in electrical and computer engineering, which is from a pure engineering point of view.

Let’s first think of Taylor’s series expansion;

\[ f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \cdots \]  \hspace{1cm} (2.1.1)

Where \( f'(0) = \left. \frac{df(t)}{dt} \right|_{t=0} \) and \( f''(0) = \left. \frac{d^2f(t)}{dt^2} \right|_{t=0} \)

If we look at the components of equation 2.1.1, the first one is a constant, the second one is a ramp function (in form of \( y(t) = at \)), the third one is a parabolic function and so on. Now let us consider a function

\[ f(t) = 10 + 3\cos(\omega_0t) + 5\cos(2\omega_0t + 30^\circ) + 4\sin(3\omega_0t) \]
This signal is of course periodic with $T = \frac{2\pi}{w_o}$.

If we use Euler’s relation, which is $e^{j\omega t} = \cos(\omega t) \pm \sin(\omega t)$;

Our function $f(t)$ becomes,

$$f(t) = 10 + \frac{3}{2} [e^{j\omega t} + e^{-j\omega t}] + \frac{5}{2} [e^{j(2\omega t + 30^\circ)} + e^{-j(2\omega t + 30^\circ)}] + \frac{4}{2}j [e^{j3\omega t} + e^{-j3\omega t}]$$

or

$$f(t) = \left(2e^{j\pi/2}\right) e^{-j3\omega t} + \left(2.5e^{-j\pi/6}\right) e^{-j2\omega t} + 1.5e^{-j\omega t}$$

$$+ 10$$

$$+ 1.5e^{j\omega t} + \left(2.5e^{j\pi/6}\right) e^{j2\omega t} + \left(2e^{-j\pi/2}\right) e^{j3\omega t}$$

Let

$$f(t) = c_{-3}e^{-j3\omega t} + c_{-2}e^{-j2\omega t} + c_{-1}e^{-j\omega t}$$

$$+ c_0$$

$$+ c_1e^{j\omega t} + c_2e^{j2\omega t} + c_3e^{j3\omega t}$$

hence our function $f(t)$ can be re-written as

$$f(t) = \sum_{k=-3}^{3} c_k e^{jk\omega t}$$

Thanks to Euler’s relation; it enabled us to convert a function that is sum of sinusiodals
into another form that is sum of complex exponential functions.

**Definition**

Given a periodic signal $f(t)$, a harmonic series for this signal is defined as [3]

$$
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0t} \quad \text{and} \quad c_k = c^*_k \quad \text{(2.1.2)}
$$

where $w_0$ = fundamental frequency, or the first harmonic

$k w_0 = k^{th}$ component

This harmonic series is called *Fourier series*, and the coefficients, $c_k$ are called as *Fourier coefficients*. The representation in equation 2.1.2 is called as the **exponential form of Fourier series**. The Fourier coefficients in general are complex numbers. Hence, we can write these coefficients in exponential form, such as

$$c_k = \|c_k\| e^{j\theta_k}
$$

Remember, in equation 2.1.2, we noted that $c_k = c^*_k$; so one can conclude that $\theta_{-k} = -\theta_k$, because, again, from equation 2.1.2, we know that $c_{-k} = c^*_k$.

For a given value of $k$, the sum of the two terms of the same frequency $kw_0$ is,

$$c_{-k} e^{-jkw_0t} + c_k e^{jkw_0t} = \|c_{-k}\| e^{-j\theta_k} e^{-jkw_0t} + \|c_k\| e^{j\theta_k} e^{jkw_0t}
$$

$$= \|c_k\| e^{j(kw_0t + \theta_k)} + e^{j(kw_0t - \theta_k)}
$$

$$= 2 \|c_k\| \cos (kw_0t + \theta_k)
$$

In the above equation, we used Euler’s identity and the property $\|c_{-k}\| = \|c_k\|$.

Hence, given the Fourier coefficients $c_k$, the **combined trigonometric form of the Fourier series** is
Recall the trigonometric identity, \( \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \) and use it for the equation 2.1.3, we get,

\[
f(t) = c_o + \sum_{k=1}^{\infty} 2\|c_k\| \cos(\theta_k t) \cos(kw_o t + \theta_k) \cos(k w_o t) - 2\|c_k\| \sin(\theta_k) \sin(kw_o t) \sin(kw_o t) \]  

(2.1.4)

Remember we have noted that \( c_k = \|c_k\| e^{j\theta_k} \), hence \( 2\|c_k\| e^{j\theta_k} = 2\|c_k\| \cos(\theta_k) + j2\|c_k\| \sin(\theta_k) \). This is in the form of \( a_k + jb_k \); so, similarly, in equation 2.1.4, let

\[
c_o = a_o, \quad 2\|c_k\| \cos(\theta_k) = a_k, \quad \text{and} \quad -2\|c_k\| \sin(\theta_k) = b_k
\]

now, equation 2.1.4 becomes

\[
f(t) = a_o + \sum_{k=1}^{\infty} [a_k \cos(kw_o t) + b_k \sin(kw_o t)]
\]

(2.1.5)

Equation 2.1.5 is the trigonometric form of Fourier series.

Now we have seen Fourier series in three forms: Exponential, combined trigonometric, and trigonometric forms.

**Fourier Coefficients**

Remember that \( f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_o t} \), and \( f(t) \) is periodic with the fundamental period \( T_o = \frac{2\pi}{w_o} \). If we multiply both sides of \( f(t) \), and integrate from \( t_o \) to \( t_o + T_o \), where \( t_o \) is just a constant (let’s assume \( t_o = 0 \) for this case) our equation turns out to be

26
\[ \int_0^{T_o} f(t)e^{-jnw_o t} \, dt = \int_0^{T_o} \left[ \sum_{k=-\infty}^{\infty} c_k e^{jkw_o t} \right] e^{jnw_o t} \, dt \]
\[ = \sum_{k=-\infty}^{\infty} c_k \int_0^{T_o} e^{j(k-n)w_o t} \, dt \]

Using Euler’s relation,

\[ c_k \int_0^{T_o} e^{j(k-n)w_o t} \, dt = c_k \int_0^{T_o} \cos [(k-n)w_o t] \, dt + jc_k \int_0^{T_o} \sin [(k-n)w_o t] \, dt \]

The integration involving the sine term equals to zero since we are integrating a sine function over an integer number of periods. The same is valid for the integration of cosine term except when \( k = n \), and for this case,

\[
\left. c_k \int_0^{T_o} \cos [(k-n)w_o t] \, dt \right|_{k=n} = c_n \int_0^{T_o} \, dt = c_n T_o
\]

hence,

\[ \int_0^{T_o} f(t)e^{-jnw_o t} \, dt = c_n T_o \]

and, this leads to

\[ c_k = \frac{1}{T_o} \int_0^{T_o} f(t)e^{-jkw_o t} \, dt \]

27
more generally,

\[ c_k = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} f(t) e^{-jkw_o t} dt \]

Note now that \( c_0 = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} f(t) dt \), which is the *average value* of the function \( f(t) \). This value is also referred to as the *dc value*.

If we check back with vector spaces, we can see that all functions that are periodic with a period of \( T_o \) form a vector space. Now, if we look at the exponential form of the Fourier series, we easily can see that \( \{ e^{jnw_o t} \}_{n=-\infty}^{\infty} \) form a basis for this space. The exponential Fourier coefficients \( \{ c_n \}_{n=-\infty}^{\infty} \) represent a given function \( f(t) \) by using this very basis set. In other words, they are the weighting coefficients, that determine the weights of the components of \( f(t) \) along each of the basis vectors. Hence, just having the coefficients \( \{ c_n \}_{n=-\infty}^{\infty} \), we can analyze the function these coefficients construct.

When talking about the orthonormal basis sets, we saw how we could find out the coefficients that construct a vector in a space \( V \) by taking the linear combinations of the basis vectors. We also realized that the easiest way to find the coefficients was to take an inner product of the vector with the orthonormal basis vectors. Now, let us see if the Fourier coefficients can be found out by using the same approach.

Assume we have two functions \( f(t) \) and \( g(t) \) in the same vector space, created by \( \{ e^{jnw_o t} \}_{n=-\infty}^{\infty} \). Then the inner product of \( f(t) \) and \( g(t) \) is defined to be

\[ f(t) \cdot g(t) = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} f(t) g(t)^* dt \quad (2.1.6) \]

hence
\[ e^{jmw_o t} \cdot e^{jnw_o t} = \frac{1}{T_o} \int_0^{T_o} e^{jmw_o t} e^{-jnw_o t} dt \]
\[ = \frac{1}{T_o} \int_0^{T_o} e^{j(m-n)w_o t} dt \]

we easily can see that,

\[ e^{jmw_o t} \cdot e^{jnw_o t} = 1 \quad \text{when} \ m = n \]

and when \( m \neq n \),

\[ e^{jmw_o t} \cdot e^{jnw_o t} = \frac{1}{T_o} \int_0^{T_o} e^{j(m-n)w_o t} dt \]
\[ = \frac{1}{T_o} \left. e^{j(m-n)w_o t} \right|_{t=0}^{T_o} \]
\[ = \frac{1}{T_o j(m-n)w_o} \left( e^{j(m-n)2\pi T_o} - e^0 \right) \]
\[ = \frac{1}{j(m-n)2\pi} \left( e^{j(m-n)2\pi} - 1 \right) \]

since \( m \) and \( n \) are integers, then \( m - n = k \) is an integer too.

Hence,

\[ e^{jmw_o t} \cdot e^{jnw_o t} = \frac{1}{jk2\pi} \left( e^{jk2\pi} - 1 \right) \]
\[ = \frac{1}{jk2\pi} (\cos (k2\pi) + j \sin (k2\pi) - 1) \]
\[
= \frac{1}{jk2\pi}(1 + 0 - 1) \\
= 0
\]

Now we conclude that the basis set \( \{e^{jn\omega_0t}\}^\infty_{n=-\infty} \) is orthonormal [4]. If we use this fact, we can calculate the Fourier coefficients by taking the inner product of \( f(t) \) and basis vectors \( e^{jn\omega_0t} \), like

\[
c_n = f(t) \cdot e^{jn\omega_0t} = \frac{1}{T_0} \int_0^{T_0} f(t)e^{-jn\omega_0t} dt \tag{2.1.7}
\]

It will be worthwhile to mention at this point that the basis set \( \{e^{jn\omega_0t}\}^\infty_{n=-\infty} \) is also repetitions of the set that is constituted by the \( n \)-th roots of unity. Notice, however, if we had Discrete Time Fourier Transform, then we would have the \( n \)-th roots of unity, but also the primitive \( n \)-th root of unity as well. Hence, sampling is dealing with the \( n \)-th roots of unity, and the primitive \( n \)-th root of unity.
References


